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Oscillations of the mixed pseudo–Dirac neutrinos*

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Abstract

Oscillations of three pseudo–Dirac flavor neutrinos ν_e, ν_μ, ν_τ are considered: $0 < m^{(L)} = m^{(R)} \ll m^{(D)}$ for their Majorana and Dirac masses taken as universal before family mixing. The actual neutrino mass matrix is assumed to be the tensor product $M^{(\nu)} \otimes \begin{pmatrix} \lambda^{(L)} & 1 \\ 1 & \lambda^{(R)} \end{pmatrix}$, where $M^{(\nu)}$ is a neutrino family mass matrix ($M^{(\nu)\,\dagger} = M^{(\nu)}$) and $\lambda^{(L,R)} = m^{(L,R)}/m^{(D)}$. The $M^{(\nu)}$ is tried in a form proposed previously for charged leptons e, μ, τ for which it gives $m_\tau = 1776.80$ MeV *versus* $m_\tau^{\rm exp} = 1777.05^{+0.29}_{-0.20}$ MeV (with the experimental values of m_e and m_μ used as inputs). However, in contrast to the charged–lepton case, in the neutrino case its off–diagonal entries dominate over diagonal. Then, it is shown that three neutrino effects (the deficits of solar ν_e's and atmospheric ν_μ's as well as the possible LSND excess of ν_e's in accelerator ν_μ beam) can be explained by neutrino oscillations though, alternatively, the LSND effect may be eliminated (by a parameter choice). Atmospheric ν_μ's oscillate dominantly into ν_τ's, while solar ν_e's — into (automatically existing) Majorana sterile counterparts of ν_e's.

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Let us consider three flavor neutrinos ν_e, ν_μ, ν_τ and assume for them the mass matrix in the form of tensor product of the neutrino family 3×3 mass matrix $\left(M^{(\nu)}_{\alpha\beta}\right)$ $(\alpha, \beta = e, \mu, \tau)$ and the Majorana 2×2 mass matrix

$$\left(\begin{array}{cc} m^{(L)} & m^{(D)} \\ m^{(D)} & m^{(R)} \end{array}\right) \;, \tag{1}$$

the second divided by $m^{(D)}$ (with $m^{(D)}$ included into $M^{(\nu)}_{\alpha\beta}$). Then, the neutrino mass term in the lagrangian gets the form

$$\begin{aligned} -\mathcal{L}_{\rm mass} &= \frac{1}{2}\sum_{\alpha\beta}\left(\overline{\overset{\circ}{\nu}}{}^{(a)}_\alpha \,,\, \overline{\overset{\circ}{\nu}}{}^{(s)}_\alpha\right) M^{(\nu)}_{\alpha\beta}\left(\begin{array}{cc} \lambda^{(L)} & 1 \\ 1 & \lambda^{(R)} \end{array}\right)\left(\begin{array}{c} \overset{\circ}{\nu}{}^{(a)}_\beta \\ \overset{\circ}{\nu}{}^{(s)}_\beta \end{array}\right) \\ &= \frac{1}{2}\sum_{\alpha\beta}\left(\overline{\left(\overset{\circ}{\nu}_{\alpha L}\right)^c} \,,\, \overline{\overset{\circ}{\nu}_{\alpha R}}\right) M^{(\nu)}_{\alpha\beta}\left(\begin{array}{cc} \lambda^{(L)} & 1 \\ 1 & \lambda^{(R)} \end{array}\right)\left(\begin{array}{c} \overset{\circ}{\nu}_{\beta L} \\ \left(\overset{\circ}{\nu}_{\beta R}\right)^c \end{array}\right) + {\rm h.c.}\;, \end{aligned} \tag{2}$$

where

$$\overset{\circ}{\nu}{}^{(a)}_\alpha \equiv \overset{\circ}{\nu}_{\alpha\,L} + \left(\overset{\circ}{\nu}_{\alpha\,L}\right)^c \;,\; \overset{\circ}{\nu}{}^{(s)}_\alpha \equiv \overset{\circ}{\nu}_{\alpha\,R} + \left(\overset{\circ}{\nu}_{\alpha\,R}\right)^c \tag{3}$$

and $\lambda^{(L,R)} \equiv m^{(L,R)}/m^{(D)}$. Here, $\overset{\circ}{\nu}{}^{(a)}_\alpha$ and $\overset{\circ}{\nu}{}^{(s)}_\alpha$ are the conventional Majorana active and sterile neutrinos of three families as they appear in the lagrangian before diagonalization of neutrino and charged–lepton family mass matrices. Due to the relation $\overline{\nu^c_\alpha}\nu_\beta = \overline{\nu^c_\beta}\nu_\alpha$, the family mass matrix $M^{(\nu)} = M^{(\nu)\,\dagger}$, when standing at the position of $\lambda^{(L)}$ and $\lambda^{(R)}$ in Eq. (2), reduces to its symmetric part $\frac{1}{2}(M^{(\nu)} + M^{(\nu)\,T})$ equal to its real part $\frac{1}{2}(M^{(\nu)} + M^{(\nu)\,*}) = {\rm Re}\,M^{(\nu)}$. We will simply assume that (at least approximately) $M^{(\nu)} = M^{(\nu)\,T} = M^{(\nu)\,*}$, and hence $U^{(\nu)} = U^{(\nu)\,*} = \left(U^{(\nu)\,-1}\right)^T$. Then, CP violation for neutrinos does not appear if, in addition, $U^{(e)} = U^{(e)\,*}$. Further on, we will always assume that $0 < \lambda^{(L)} = \lambda^{(R)}$ $(\equiv \lambda^{(M)})$ and $\lambda^{(M)} \ll 1$ (the pseudo–Dirac case) [1].

Then, diagonalizing the neutrino mass matrix, we obtain from Eq. (2)

$$-\mathcal{L}_{\rm mass} = \frac{1}{2}\sum_i \left(\overline{\nu}^I_i \,,\, \overline{\nu}^{II}_i\right) m_{\nu_i}\left(\begin{array}{cc} \lambda^I & 0 \\ 0 & \lambda^{II} \end{array}\right)\left(\begin{array}{c} \nu^I_i \\ \nu^{II}_i \end{array}\right) \;, \tag{4}$$

where

$$\left(U^{(\nu)\,\dagger}\right)_{i\,\alpha} M^{(\nu)}_{\alpha\beta} U^{(\nu)}_{\beta\,j} = m_{\nu_i}\delta_{ij} \quad , \quad \lambda^{I,\,II} = \mp 1 + \lambda^{(M)} \simeq \mp 1 \tag{5}$$

$(i,\, j = 1, 2, 3)$ and

$$\nu^{I,\,II}_i = \sum_i \left(U^{(\nu)\,\dagger}\right)_{i\,\alpha} \frac{1}{\sqrt{2}} \left(\overset{\circ}{\nu}{}^{(a)}_\alpha \mp \overset{\circ}{\nu}{}^{(s)}_\alpha\right) = \sum_i V_{i\,\alpha} \frac{1}{\sqrt{2}} \left(\nu^{(a)}_\alpha \mp \nu^{(s)}_\alpha\right) \tag{6}$$

with $V_{i\,\alpha} = \left(U^{(\nu)\,\dagger}\right)_{i\,\beta} U^{(e)}_{\beta\alpha}$ describing the lepton counterpart of the Cabibbo—Kobayashi—Maskawa matrix. Here,

$$\nu^{(a,s)}_\alpha \equiv \sum_\beta \left(U^{(e)\,\dagger}\right)_{\alpha\beta} \overset{\circ}{\nu}{}^{(a,s)}_\beta = \sum_i \left(V^\dagger\right)_{\alpha\,i} \frac{1}{\sqrt{2}} \left(\pm\nu^I_i + \nu^{II}_i\right) = \nu_{\alpha\,L,R} + \left(\nu_{\alpha\,L,R}\right)^c \tag{7}$$

and

$$\left(U^{(e)\,\dagger}\right)_{\alpha\gamma} M^{(e)}_{\gamma\delta} U^{(e)}_{\delta\beta} = m_{e_\alpha}\delta_{\alpha\beta}\ , \tag{8}$$

where $\left(M^{(e)}_{\alpha\beta}\right)$ $(\alpha, \beta = e,\, \mu,\, \tau)$ is the mass matrix for three charged leptons e^-, μ^-, τ^-, giving their masses m_e, m_μ, m_τ after its diagonalization is carried out. Now, $\nu^{(a)}_\alpha$ and $\nu^{(s)}_\alpha$ are the conventional Majorana active and sterile flavor neutrinos of three families, while ν^I_i and ν^{II}_i are Majorana massive neutrinos.

If CP violation for neutrinos does not appear or can be neglected, the probabilities for oscillations $\nu^{(a)}_\alpha \to \nu^{(a)}_\beta$ and $\nu^{(a)}_\alpha \to \nu^{(s)}_\beta$ are given by the following formulae (in the pseudo–Dirac case):

$$\begin{aligned} P\left(\nu^{(a)}_\alpha \to \nu^{(a)}_\beta\right) &= |\langle\nu^{(a)}_\beta|e^{iPL}|\nu^{(a)}_\alpha\rangle|^2 = \delta_{\beta\,\alpha} - \sum_i |V_{i\,\beta}|^2 |V_{i\,\alpha}|^2 \sin^2\left(x^{II}_i - x^I_i\right) \\ &-\sum_{j>i} V_{j\,\beta} V^*_{j\,\alpha} V^*_{i\,\beta} V_{i\,\alpha} \Big[\sin^2\left(x^I_j - x^I_i\right) + \sin^2\left(x^{II}_j - x^{II}_i\right) + \sin^2\left(x^{II}_j - x^I_i\right) + \sin^2\left(x^I_j - x^{II}_i\right)\Big] \end{aligned} \tag{9}$$

and

$$\begin{aligned} P\left(\nu^{(a)}_\alpha \to \nu^{(s)}_\beta\right) &= |\langle\nu^{(s)}_\beta|e^{iPL}|\nu^{(a)}_\alpha\rangle|^2 = \sum_i |V_{i\,\beta}|^2 |V_{i\,\alpha}|^2 \sin^2\left(x^{II}_i - x^I_i\right) \\ &-\sum_{j>i} V_{j\,\beta} V^*_{j\,\alpha} V^*_{i\,\beta} V_{i\,\alpha} \left[\sin^2\left(x^I_j - x^I_i\right) + \sin^2\left(x^{II}_j - x^{II}_i\right) - \sin^2\left(x^{II}_j - x^I_i\right) - \sin^2\left(x^I_j - x^{II}_i\right)\right] , \end{aligned} \tag{10}$$

where $P|\nu_i^{I,II}\rangle = p_i^{I,II}|\nu_i^{I,II}\rangle$, $p_i^{I,II} = \sqrt{E^2 - (m_{\nu_i}\lambda^{I,II})^2} \simeq E - (m_{\nu_i}\lambda^{I,II})^2/2E$ and

$$x_i^{I,II} = 1.27\frac{(m_{\nu_i}^2\lambda^{I,II})^2 L}{E} \;\;,\;\; (\lambda^{I,II})^2 = 1 \mp 2\lambda^{(M)} \simeq 1 \tag{11}$$

with m_{ν_i}, L and E expressed in eV, km and GeV, respectively (L is the experimental baseline). Here, due to Eqs. (11),

$$x_i^{II} - x_i^{I} = 1.27\frac{4m_{\nu_i}^2\lambda^{(M)}L}{E} \tag{12}$$

and for $j > i$

$$x_j^{I} - x_i^{I} \simeq x_j^{II} - x_i^{II} \simeq x_j^{II} - x_i^{I} \simeq x_j^{I} - x_i^{II} \simeq 1.27\frac{(m_{\nu_j}^2 - m_{\nu_i}^2)L}{E}\;. \tag{13}$$

Then, the bracket [ ] in Eq. (9) and (10) is reduced to $4\sin^2 1.27(m_{\nu_j}^2 - m_{\nu_i}^2)L/E$ and 0, respectively. The probability sum rule $\sum_\beta \left[P\left(\nu_\alpha^{(a)} \to \nu_\beta^{(a)}\right) + P\left(\nu_\alpha^{(a)} \to \nu_\beta^{(s)}\right)\right] = 1$ follows readily from Eqs. (9) and (10).

Notice that in the case of lepton Cabibbo—Kobayashi—Maskawa matrix being nearly unit, $(V_{i\,\alpha}) \simeq (\delta_{i\,\alpha})$, the oscillations $\nu_\alpha^{(a)} \to \nu_\beta^{(a)}$ and $\nu_\alpha^{(a)} \to \nu_\beta^{(s)}$ are essentially described by the formulae

$$\begin{aligned} P\left(\nu_\alpha^{(a)} \to \nu_\beta^{(a)}\right) &\simeq \delta_{\beta\alpha} - P\left(\nu_\alpha^{(a)} \to \nu_\beta^{(s)}\right)\;, \\ P\left(\nu_\alpha^{(a)} \to \nu_\beta^{(s)}\right) &\simeq \delta_{\beta\alpha}\sin^2\left(1.27\frac{4m_{\nu_\alpha}^2\lambda^{(M)}L}{E}\right) \end{aligned} \tag{14}$$

corresponding to three maximal mixings of $\nu_\alpha^{(a)}$ with $\nu_\alpha^{(s)}$ ($\alpha = e,\, \mu,\, \tau$). Of course, for a further discussion of the oscillation formulae (9) and (10), in particular those for appearance modes $\nu_\alpha^{(a)} \to \nu_\beta^{(a)}$ ($\alpha \neq \beta$), a detailed knowledge of $(V_{i\,\alpha})$ is necessary.

To this end, we will try to extend to neutrinos the form of charged–lepton mass matrix

$$\left(M_{\alpha\beta}^{(e)}\right) = \frac{1}{29}\begin{pmatrix} \mu^{(e)}\varepsilon^{(e)} & 2\alpha^{(e)}e^{i\varphi^{(e)}} & 0 \\ 2\alpha^{(e)}e^{-i\varphi^{(e)}} & 4\mu^{(e)}(80+\varepsilon^{(e)})/9 & 8\sqrt{3}\,\alpha^{(e)}e^{i\varphi^{(e)}} \\ 0 & 8\sqrt{3}\,\alpha^{(e)}e^{-i\varphi^{(e)}} & 24\mu^{(e)}(624+\varepsilon^{(e)})/25 \end{pmatrix} \tag{15}$$

which reproduces surprisingly well the charged–lepton masses m_e, m_μ, m_τ ($\mu^{(e)}$, $\alpha^{(e)}$ and $\varepsilon^{(e)}$ are positive parameters). In fact, treating off–diagonal elements of $\left(M^{(e)}_{\alpha\beta}\right)$ as a perturbation of its diagonal entries, we get the mass sum rule

$$
\begin{aligned}
m_\tau &= \frac{6}{125}\left(351 m_\mu - 136 m_e\right) + 10.2112\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2 \ \mathrm{MeV} \\
&= \left[1776.80 + 10.2112\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right] \ \mathrm{MeV}\ ,
\end{aligned}
\tag{16}
$$

where the experimental values of m_e and m_μ are used as inputs. Then, $\mu^{(e)} = 85.9924$ MeV and $\varepsilon^{(e)} = 0.172329$ (up to the perturbation). The prediction (16) agrees very well with the experimental figure $m_\tau^{\mathrm{exp}} = 1777.05^{+0.29}_{-0.20}$ MeV, even in the zero order in $\left(\alpha^{(e)}/\mu^{(e)}\right)^2$. Taking this experimental value of m_τ as another input, we obtain

$$
\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2 = 0.024^{+0.028}_{-0.025}\ ,
\tag{17}
$$

what is not inconsistent with zero.

Now, we conjecture the neutrino family mass matrix $\left(M^{(\nu)}_{\alpha\beta}\right)$ in the form (15) with $\mu^{(e)} \to \mu^{(\nu)}$, $\alpha^{(e)} \to \alpha^{(\nu)}$, $\varepsilon^{(e)} \to \varepsilon^{(\nu)} \simeq 0$ and $\varphi^{(e)} \to \varphi^{(\nu)} = 0$ [2]. In order to get the neutrino family diagonalizing matrix $\left(U^{(\nu)}_{\alpha i}\right)$ rather different from the unit matrix $\left(\delta^{(\nu)}_{\alpha i}\right)$, we assume that diagonal elements of $\left(M^{(\nu)}_{\alpha\beta}\right)$ can be considered as a perturbation of its off–diagonal entries (though the diagonal as well as the off–diagonal elements are expected to be very small). Under this assumption we derive the unitary matrix $\left(U^{(\nu)}_{\alpha i}\right)$ of the following form :

$$
\left(U^{(\nu)}_{\alpha i}\right) = \begin{pmatrix} \frac{\sqrt{48}}{7} & -\frac{1}{7\sqrt{2}} e^{i\varphi^{(\nu)}} & \frac{1}{7\sqrt{2}} e^{2i\varphi^{(\nu)}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\, e^{i\varphi^{(\nu)}} \\ -\frac{1}{7}\, e^{-2i\varphi^{(\nu)}} & -\frac{\sqrt{48}}{7\sqrt{2}}\, e^{-i\varphi^{(\nu)}} & \frac{\sqrt{48}}{7\sqrt{2}} \end{pmatrix} + O(\xi/7)
\tag{18}
$$

with $\varphi^{(\nu)} = 0$ and

$$
\xi \equiv \frac{M^{(\nu)}_{33}}{|M^{(\nu)}_{12}|} = 299.52 \frac{\mu^{(\nu)}}{\alpha^{(\nu)}}\ \ ,\ \ \chi \equiv \frac{M^{(\nu)}_{22}}{|M^{(\nu)}_{12}|} = \frac{\xi}{16.848}\ .
\tag{19}
$$

In this case, the neutrino family masses are

$$m_{\nu_1} = \frac{\xi}{49}|M^{(\nu)}_{12}| \;\; , \;\; m_{\nu_2,\,\nu_3} = \left[\mp 7 + \frac{1}{2}\left(\frac{48}{49}\xi + \chi\right)\right]|M^{(\nu)}_{12}| \; , \tag{20}$$

where $|M^{(\nu)}_{12}| = 2\alpha^{(\nu)}/29$ (thus, $m_{\nu_1} \ll |m_{\nu_2}| < m_{\nu_3}$). Hence,

$$m^2_{\nu_3} - m^2_{\nu_2} = 14\left(\frac{48}{49}\xi + \chi\right)|M^{(\nu)}_{12}|^2 = 20.721\,\alpha^{(\nu)}\mu^{(\nu)} \; . \tag{21}$$

Taking in contrast $\left(U^{(e)}_{\alpha\beta}\right) \simeq (\delta_{\alpha\beta})$ — as in $\left(M^{(e)}_{\alpha\beta}\right)$ the off–diagonal elements are perturbatively small *versus* diagonal entries [*cf.* Eq. (17)] — we can insert

$$V_{i\,\alpha} \simeq \left(U^{(\nu)\,\dagger}\right)_{i\,\alpha} = U^{(\nu)\,*}_{\alpha\,i} \tag{22}$$

into Eqs. (9) and (10). Here, $U^{(\nu)\,*}_{\alpha\,i}$ are determined from Eq. (18).

Then, with the use of Eqs. (12) and (13) the $\nu^{(a)}_\alpha \to \nu^{(a)}_\beta$ oscillation formulae (9) take the form

$$\begin{aligned}
P\left(\nu^{(a)}_e \to \nu^{(a)}_e\right) &= 1 - \frac{48^2}{49^2}\sin^2\left(1.27\frac{4m^2_{\nu_1}\lambda^{(M)}L}{E}\right) \\
&\quad -\frac{1}{4\cdot 49^2}\left[\sin^2\left(1.27\frac{4m^2_{\nu_2}\lambda^{(M)}L}{E}\right) + \sin^2\left(1.27\frac{4m^2_{\nu_3}\lambda^{(M)}L}{E}\right)\right] \\
&\quad -\frac{96}{49^2}\left[\sin^2\left(1.27\frac{(m^2_{\nu_2} - m^2_{\nu_1})L}{E}\right) + \sin^2\left(1.27\frac{(m^2_{\nu_3} - m^2_{\nu_1})L}{E}\right)\right] \\
&\quad -\frac{1}{49^2}\sin^2\left(1.27\frac{(m^2_{\nu_3} - m^2_{\nu_2})L}{E}\right) \; , \\
P\left(\nu^{(a)}_\mu \to \nu^{(a)}_\mu\right) &= 1 - \frac{1}{4}\left[\sin^2\left(1.27\frac{4m^2_{\nu_2}\lambda^{(M)}L}{E}\right) + \sin^2\left(1.27\frac{4m^2_{\nu_3}\lambda^{(M)}L}{E}\right)\right] \\
&\quad - \sin^2\left(1.27\frac{(m^2_{\nu_3} - m^2_{\nu_2})L}{E}\right) \; , \\
P\left(\nu^{(a)}_\mu \to \nu^{(a)}_e\right) &= -\frac{1}{4\cdot 49}\left[\sin^2\left(1.27\frac{4m^2_{\nu_2}\lambda^{(M)}L}{E}\right) + \sin^2\left(1.27\frac{4m^2_{\nu_3}\lambda^{(M)}L}{E}\right)\right] \\
&\quad +\frac{1}{49}\sin^2\left(1.27\frac{(m^2_{\nu_3} - m^2_{\nu_2})L}{E}\right)
\end{aligned} \tag{23}$$

and

$$
\begin{aligned}
P\left(\nu_\mu^{(a)} \to \nu_\tau^{(a)}\right) &= -\frac{48}{4\cdot 49}\left[\sin^2\left(1.27\frac{4m_{\nu_2}^2\lambda^{(M)}L}{E}\right) + \sin^2\left(1.27\frac{4m_{\nu_3}^2\lambda^{(M)}L}{E}\right)\right] \\
&\quad +\frac{48}{49}\sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L}{E}\right) \;, \\
P\left(\nu_e^{(a)} \to \nu_\tau^{(a)}\right) &= -\frac{48}{49^2}\sin^2\left(1.27\frac{4m_{\nu_1}^2\lambda^{(M)}L}{E}\right) \\
&\quad -\frac{48}{4\cdot 49^2}\left[\sin^2\left(1.27\frac{4m_{\nu_2}^2\lambda^{(M)}L}{E}\right) + \sin^2\left(1.27\frac{4m_{\nu_3}^2\lambda^{(M)}L}{E}\right)\right] \\
&\quad +\frac{96}{49^2}\left[\sin^2\left(1.27\frac{(m_{\nu_2}^2 - m_{\nu_1}^2)L}{E}\right) + \sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_1}^2)L}{E}\right)\right] \\
&\quad -\frac{48}{49^2}\sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L}{E}\right) \;, \\
P\left(\nu_\tau^{(a)} \to \nu_\tau^{(a)}\right) &= 1-\frac{1}{49^2}\sin^2\left(1.27\frac{4m_{\nu_1}^2\lambda^{(M)}L}{E}\right) \\
&\quad -\frac{48^2}{4\cdot 49^2}\left[\sin^2\left(1.27\frac{4m_{\nu_2}^2\lambda^{(M)}L}{E}\right) + \sin^2\left(1.27\frac{4m_{\nu_3}^2\lambda^{(M)}L}{E}\right)\right] \\
&\quad -\frac{96}{49^2}\left[\sin^2\left(1.27\frac{(m_{\nu_2}^2 - m_{\nu_1}^2)L}{E}\right) + \sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_1}^2)L}{E}\right)\right] \\
&\quad -\frac{48^2}{49^2}\sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L}{E}\right) \;. \qquad (24)
\end{aligned}
$$

In these formulae, the experimental baselines L (and neutrino energies E) are generally different.

Further on, we intend to relate the first, second and third Eq. (23) to the experimental results concerning the deficit of solar ν_e's [3], the deficit of atmospheric ν_μ's [4] and the excess of ν_e's in accelerator ν_μ beam [5], respectively.

First, let us assume the simplifying hypothesis that the LSND effect [5] does not exist. Then, under the numerical conjecture that

$$
\begin{aligned}
1.27\frac{4m_{\nu_1}^2\lambda^{(M)}L_{\rm sol}}{E_{\rm sol}} = O(1) \;\;&,\;\; 1.27\frac{4m_{\nu_2}^2\lambda^{(M)}L_{\rm atm}}{E_{\rm atm}} = O\left(\frac{m_{\nu_2}^2 L_{\rm atm}/E_{\rm atm}}{m_{\nu_1}^2 L_{\rm sol}/E_{\rm sol}}\right) \ll 1\,, \\
1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L_{\rm atm}}{E_{\rm atm}} = O(1) \;\;&,\;\; 1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L_{\rm sol}}{E_{\rm sol}} = O\left(\frac{L_{\rm sol}/E_{\rm sol}}{L_{\rm atm}/E_{\rm atm}}\right) \gg 1\,, \qquad (25)
\end{aligned}
$$

we obtain from Eqs. (23)

$$
\begin{aligned}
P\left(\nu_e^{(a)} \to \nu_e^{(a)}\right) &\simeq 1 - \frac{48^2}{49^2}\sin^2\left(1.27\frac{4m_{\nu_1}^2\lambda^{(M)}L_{\rm sol}}{E_{\rm sol}}\right) - \frac{387}{4\cdot 49^2} \\
&\simeq 1 - \frac{48^2}{49^2}\sin^2\left(1.27\frac{4m_{\nu_1}^2\lambda^{(M)}L_{\rm sol}}{E_{\rm sol}}\right)\ , \\
P\left(\nu_\mu^{(a)} \to \nu_\mu^{(a)}\right) &\simeq 1 - \sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L_{\rm atm}}{E_{\rm atm}}\right) - 8(1.27)^2\frac{m_{\nu_2}^4\lambda^{(M)\,2}L_{\rm atm}^2}{E_{\rm atm}^2} \\
&\simeq 1 - \sin^2\left(1.27\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)L_{\rm atm}}{E_{\rm atm}}\right)\ , \\
P\left(\nu_\mu^{(a)} \to \nu_e^{(a)}\right) &\simeq -\frac{8}{49}(1.27)^2\frac{m_{\nu_2}^4\lambda^{(M)\,2}L_{\rm LSND}^2}{E_{\rm LSND}^2} + \frac{1}{49}(1.27)^2\frac{(m_{\nu_3}^2 - m_{\nu_2}^2)^2L_{\rm LSND}^2}{E_{\rm LSND}^2} \\
&\simeq 0\ . \qquad (26)
\end{aligned}
$$

The term $-387/4\cdot 49^2 = -0.0403$ in the first Eq. (26) comes out from averaging all $\sin^2$ of large phases over oscillation lengths defined by $\sin^2$ of a phase $= O(1)$ (then, each $\sin^2$ of a large phase gives 1/2).

Comparing Eqs. (26) with experimental estimates, we get for solar ν_e's [3] (using the global vacuum fit)

$$
\frac{48^2}{49^2} \leftrightarrow \sin^2 2\theta_{\rm sol} \sim 0.75\ \ ,\ \ 4m_{\nu_1}^2\lambda^{(M)} \leftrightarrow \Delta m_{\rm sol}^2 \sim 6.5\times 10^{-11}\ {\rm eV}^2\ , \qquad (27)
$$

and for atmospheric ν_μ's [4]

$$
1 \leftrightarrow \sin^2 2\theta_{\rm atm} \sim 1\ \ ,\ \ m_{\nu_3}^2 - m_{\nu_2}^2 \leftrightarrow \Delta m_{\rm atm}^2 \sim 2.2\times 10^{-3}\ {\rm eV}^2\ . \qquad (28)
$$

Thus, from Eqs. (27) and (28)

$$
\frac{4m_{\nu_1}^2\lambda^{(M)}}{m_{\nu_3}^2 - m_{\nu_2}^2} \leftrightarrow \frac{\Delta m_{\rm sol}^2}{\Delta m_{\rm atm}^2} \sim 3.0\times 10^{-8}\ . \qquad (29)
$$

Hence, making use of Eqs. (20) and (21), we infer that

$$
\begin{aligned}
\xi\lambda^{(M)} &\sim 2.6\times 10^{-4} \quad , \quad \frac{m_{\nu_1}}{|m_{\nu_2}|}\lambda^{(M)} = \frac{1}{7^3}\xi\lambda^{(M)} \sim 7.5\times 10^{-7}\ \ , \\
\frac{\mu^{(\nu)}}{\alpha^{(\nu)}}\lambda^{(M)} = \frac{1}{299.52}\xi\lambda^{(M)} &\sim 8.6\times 10^{-7} \quad , \quad \alpha^{(\nu)}\mu^{(\nu)} = \frac{m_{\nu_3}^2 - m_{\nu_2}^2}{20.721} \sim 1.1\times 10^{-4}\ {\rm eV}^2\ \ , \\
\mu^{(\nu)\,2}\lambda^{(M)} &\sim 9.1\times 10^{-11}\ {\rm eV}^2 \quad , \quad \frac{\alpha^{(\nu)\,2}}{\lambda^{(M)}} \sim 1.2\times 10^2\ {\rm eV}^2\ . \qquad (30)
\end{aligned}
$$

Here, the constant ξ still may be treated as a free parameter (determining $\lambda^{(M)}$). If $\xi = O(10^{-1})$, then $\lambda^{(M)} = O(10^{-3})$, $m_{\nu_1}/|m_{\nu_2}| = O(10^{-4})$, $\mu^{(\nu)}/\alpha^{(\nu)} = O(10^{-4})$, $\mu^{(\nu)} = O(10^{-4}\ \mathrm{eV})$, $\alpha^{(\nu)} = O(1\ \mathrm{eV})$ and

$$m_{\nu_1} = O(10^{-4}\ \mathrm{eV}) \ \ , \ \ |m_{\nu_2}| = O(10^{-1}\ \mathrm{eV}) \ \ , \ \ m_{\nu_3} = O(10^{-1}\ \mathrm{eV}) \tag{31}$$

with $m^2_{\nu_3} - m^2_{\nu_2} \sim 2.2 \times 10^{-3}\ \mathrm{eV}^2$.

In this way, both neutrino deficits can be explained by pseudo–Dirac neutrino oscillations. Note that solar $\nu_e^{(a)}$'s and atmospheric $\nu_\mu^{(a)}$'s oscillate dominantly into $\nu_e^{(s)}$'s and $\nu_\tau^{(a)}$'s, respectively (here, $\nu^{(a)}_{\alpha\, L} = \nu_{\alpha\, L}$, $\nu^{(s)}_{\alpha\, L} = (\nu^c_\alpha)_L$).

Now, let us accept the LSND effect [5]. Then, making the numerical conjecture that

$$\begin{array}{rcl}
1.27\dfrac{4m^2_{\nu_1}\lambda^{(M)}L_{\mathrm{sol}}}{E_{\mathrm{sol}}} = O(1) \ \ , \ \ 1.27\dfrac{4m^2_{\nu_2}\lambda^{(M)}L_{\mathrm{atm}}}{E_{\mathrm{atm}}} & = & O\left(\dfrac{m^2_{\nu_2}L_{\mathrm{atm}}/E_{\mathrm{atm}}}{m^2_{\nu_1}L_{\mathrm{sol}}/E_{\mathrm{sol}}}\right) < 1\,, \\
1.27\dfrac{(m^2_{\nu_3}-m^2_{\nu_2})L_{\mathrm{LSND}}}{E_{\mathrm{LSND}}} = O(1) \ \ , \ 1.27\dfrac{(m^2_{\nu_3}-m^2_{\nu_2})L_{\mathrm{atm}}}{E_{\mathrm{atm}}} & = & O\left(\dfrac{L_{\mathrm{atm}}/E_{\mathrm{atm}}}{L_{\mathrm{LSND}}/E_{\mathrm{LSND}}}\right) \gg 1\,,
\end{array} \tag{32}$$

we get from Eqs. (23)

$$\begin{array}{rcl}
P\left(\nu_e^{(a)} \to \nu_e^{(a)}\right) & \simeq & 1 - \dfrac{48^2}{49^2}\sin^2\left(1.27\dfrac{4m^2_{\nu_1}\lambda^{(M)}L_{\mathrm{sol}}}{E_{\mathrm{sol}}}\right) - \dfrac{387}{4\cdot 49^2} \\
& \simeq & 1 - \dfrac{48^2}{49^2}\sin^2\left(1.27\dfrac{4m^2_{\nu_1}\lambda^{(M)}L_{\mathrm{sol}}}{E_{\mathrm{sol}}}\right)\ , \\
P\left(\nu_\mu^{(a)} \to \nu_\mu^{(a)}\right) & \simeq & 1 - \dfrac{1}{2} - \dfrac{1}{2}\sin^2\left(1.27\dfrac{4m^2_{\nu_2}\lambda^{(M)}L_{\mathrm{atm}}}{E_{\mathrm{atm}}}\right)\ , \\
P\left(\nu_\mu^{(a)} \to \nu_e^{(a)}\right) & \simeq & \dfrac{1}{49}\sin^2\left(1.27\dfrac{(m^2_{\nu_3}-m^2_{\nu_2})L_{\mathrm{LSND}}}{E_{\mathrm{LSND}}}\right) - \dfrac{8}{49}(1.27)^2\dfrac{m^4_{\nu_2}\lambda^{(M)\,2}L^2_{\mathrm{LSND}}}{E^2_{\mathrm{LSND}}} \\
& \simeq & \dfrac{1}{49}\sin^2\left(1.27\dfrac{(m^2_{\nu_3}-m^2_{\nu_2})L_{\mathrm{LSND}}}{E_{\mathrm{LSND}}}\right)\ .
\end{array} \tag{33}$$

When comparing Eqs. (33) with experimental estimates, we obtain for solar ν_e's [3] (making use of global vacuum fit)

$$\frac{48^2}{49^2} \leftrightarrow \sin^2 2\theta_{\mathrm{sol}} \sim 0.75 \ \ , \ \ 4m^2_{\nu_1}\lambda^{(M)} \leftrightarrow \Delta m^2_{\mathrm{sol}} \sim 6.5\times 10^{-11}\ \mathrm{eV}^2\ , \tag{34}$$

for atmospheric ν_μ's [4]

$$1 \leftrightarrow \sin^2 2\theta_{\rm atm} \sim 1 \;\;,\;\; \frac{1}{2} + \frac{1}{2}\sin^2\left(1.27\frac{4m^2_{\nu_2}\lambda^{(M)}L_{\rm atm}}{E_{\rm atm}}\right) \leftrightarrow \sin^2\left(1.27\frac{\Delta m^2_{\rm atm}L_{\rm atm}}{E_{\rm atm}}\right) \quad (35)$$

with

$$\Delta m^2_{\rm atm} \sim 2.2\times 10^{-3}\;{\rm eV}^2\;, \quad (36)$$

and for accelerator ν_μ's [5], say,

$$\frac{1}{49} \leftrightarrow \sin^2 2\theta_{\rm LSND} \sim 0.02 \;\;,\;\; m^2_{\nu_3} - m^2_{\nu_2} \leftrightarrow \Delta m^2_{\rm LSND} \sim 0.5\;{\rm eV}^2\;. \quad (37)$$

So, from Eqs. (34) and (37)

$$\frac{4m^2_{\nu_1}\lambda^{(M)}}{m^2_{\nu_3} - m^2_{\nu_2}} \leftrightarrow \frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm LSND}} \sim 1.3\times 10^{-10}\;. \quad (38)$$

Hence, due to Eqs. (20) and (21),

$$\begin{array}{rcl} \xi\lambda^{(M)} \sim 1.1\times 10^{-6} & , & \dfrac{m_{\nu_1}}{|m_{\nu_2}|}\lambda^{(M)} \sim 3.3\times 10^{-9}\;\;, \\ \dfrac{\mu^{(\nu)}}{\alpha^{(\nu)}}\lambda^{(M)} \sim 3.8\times 10^{-9} & , & \alpha^{(\nu)}\mu^{(\nu)} \sim 2.4\times 10^{-2}\;{\rm eV}^2\;\;, \\ \mu^{(\nu)\,2}\lambda^{(M)} \sim 9.2\times 10^{-11}\;{\rm eV}^2 & , & \dfrac{\alpha^{(\nu)\,2}}{\lambda^{(M)}} \sim 6.3\times 10^{6}\;{\rm eV}^2\;\;. \end{array} \quad (39)$$

Here, the constant ξ still may play the role of a free parameter (determining $\lambda^{(M)}$). If $\xi = O(10^{-1})$, then $\lambda^{(M)} = O(10^{-5})$, $m_{\nu_1}/|m_{\nu_2}| = O(10^{-4})$, $\mu^{(\nu)}/\alpha^{(\nu)} = O(10^{-4})$, $\mu^{(\nu)} = O(10^{-3}\;{\rm eV})$, $\alpha^{(\nu)} = O(10\;{\rm eV})$, and hence

$$m_{\nu_1} = O(10^{-3}\;{\rm eV}) \;\;,\;\; |m_{\nu_2}| = O(1\;{\rm eV}) \;\;,\;\; m_{\nu_3} = O(1\;{\rm eV}) \quad (40)$$

with $m^2_{\nu_3} - m^2_{\nu_2} \sim 0.5\;{\rm eV}^2$. Then, in Eq. (35) we can put approximately

$$\frac{1}{2} \leftrightarrow \sin^2\left(1.27\frac{\Delta m^2_{\rm atm}L_{\rm atm}}{E_{\rm atm}}\right) \simeq 1 - U/D \sim 1 - 0.54^{+0.06}_{-0.05} \quad (41)$$

in a reasonable consistency with the Super–Kamiokande estimate [4]. Here, $(U-D)/(U+D)$ is the up–down assymetry for ν_μ's, estimated as $-0.296 \pm 0.048 \pm 0.01$.

In this way, therefore, all three neutrino effects can be explained by pseudo–Dirac neutrino oscillations. Note that solar $\nu_e^{(a)}$'s and atmospheric $\nu_\mu^{(a)}$'s oscillate dominantly into $\nu_e^{(s)}$'s and $\nu_\tau^{(a)}$'s, respectively, as in the previous case when the LSND effect was absent.

The recently improved upper bound on the effective mass $\langle m_{\nu_e}\rangle$ of the Majorana $\nu_e^{(a)}$ neutrino extracted from neutrinoless double β decay experiments is 0.2 eV [6]. In our pseudo–Dirac case, this mass is given by the formula (if $V_{i\alpha} \simeq U^{(\nu)\,*}_{\alpha i}$):

$$\begin{aligned} \langle m_{\nu_e}\rangle &= |\sum_i U^{(\nu)\,2}_{e\,i} m_{\nu_i} \frac{1}{2}\left(\lambda^I + \lambda^{II}\right)| = |\sum_i U^{(\nu)\,2}_{e\,i} m_{\nu_i} \lambda^{(M)}| \\ &= \frac{1}{49 \cdot 29}\left(3 \cdot \frac{48}{49}\xi + \chi\right)\alpha^{(\nu)}\lambda^{(M)} = \frac{\xi\lambda^{(M)}}{473.96}\alpha^{(\nu)} \;, \end{aligned} \tag{42}$$

as $\varphi^{(\nu)} = 0$ in Eq. (18) (here, $U_{\alpha i} = U^*_{\alpha i}$) and $\lambda^{I,\,II} = \mp 1 + \lambda^{(M)}$, while

$$\nu_\alpha^{(a)} = \sum_i U^{(\nu)}_{\alpha i} \frac{1}{\sqrt{2}}\left(\nu_i^I + \nu_i^{II}\right) \;. \tag{43}$$

Thus, in the option excluding or accepting LSND effect we estimate from Eq. (30) or (39) that

$$\langle m_{\nu_e}\rangle \sim \left\{ \begin{array}{l} 5.4 \times 10^{-7}\alpha^{(\nu)} \sim O(10^{-6}\ \mathrm{eV}) \\ 2.4 \times 10^{-9}\alpha^{(\nu)} \sim O(10^{-8}\ \mathrm{eV}) \end{array} \right. \;, \tag{44}$$

respectively. Thus, in this pseudo–Dirac case, the $0\nu\beta\beta$ decay violating the lepton number conservation is negligible. Note that $\langle m_{\nu_e}\rangle \ll m_{\nu_1} \ll |m_{\nu_2}| < m_{\nu_3}$ in both options. Here, the neutrino masses are

$$m^{I,\,II}_{\nu_i} = m_{\nu_i}\lambda^{I,\,II} = m_{\nu_i}\left(\mp 1 + \lambda^{(M)}\right) \simeq \mp m_{\nu_i} \;. \tag{45}$$

Since for relativistic particles only masses squared are relevant, the "phenomenological" neutrino masses are equal to $|m^{I,\,II}_{\nu_i}| \simeq |m_{\nu_i}|$ *i.e.*, $\simeq m_{\nu_1}\;,\; |m_{\nu_2}|\;,\; m_{\nu_3}$.

Finally, let us turn back to the option, where there is no LSND effect. In this case, the natural possibility seems to be a (nearly) diagonal form of neutrino family mass matrix $M^{(\nu)} \simeq (\delta_{\alpha\beta}\, m_{\nu_\alpha})$ and so, unit neutrino diagonalizing matrix $U^{(\nu)} \simeq (\delta_{\alpha i})$. Then, if $U^{(e)} \simeq (\delta_{\alpha\beta})$ *i.e.*, $V \simeq (\delta_{i\alpha})$, Eqs. (14) hold, giving

$$
\begin{aligned}
P\left(\nu_e^{(a)} \to \nu_e^{(a)}\right) &\simeq 1 - P\left(\nu_e^{(a)} \to \nu_e^{(s)}\right) \simeq 1 - \sin^2\left(1.27\frac{4m^2_{\nu_e}\lambda^{(M)}L}{E}\right) \;, \\
P\left(\nu_\mu^{(a)} \to \nu_\mu^{(a)}\right) &\simeq 1 - P\left(\nu_\mu^{(a)} \to \nu_\mu^{(s)}\right) \simeq 1 - \sin^2\left(1.27\frac{4m^2_{\nu_\mu}\lambda^{(M)}L}{E}\right) \;. \qquad (46)
\end{aligned}
$$

Here, $m_{\nu_i} = m_{\nu_\alpha}$ are neutrino family masses.

Comparing Eqs. (46) with experimental estimates for solar ν_e's [3] (using the global vacuum fit) and atmospheric ν_μ's [4], we have Eq. (27) (with $m_{\nu_i} = m_{\nu_e}$) and the relation

$$
1 \leftrightarrow \sin^2 2\theta_{\rm atm} \sim 1 \;\;,\;\; 4m^2_{\nu_\mu}\lambda^{(M)} \leftrightarrow \Delta m^2_{\rm atm} \sim 2.2\times 10^{-3}\ {\rm eV}^2 \;\;, \qquad (47)
$$

respectively. Hence,

$$
\frac{m^2_{\nu_e}}{m^2_{\nu_\mu}} \leftrightarrow \frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}} \sim 3.0\times 10^{-8} \;. \qquad (48)
$$

Under the conjecture that $M^{(\nu)}$ has the form (15) with $\mu^{(e)} \to \mu^{(\nu)}$, $\alpha^{(e)} \to \alpha^{(\nu)} = 0$, $\varepsilon^{(e)} \to \varepsilon^{(\nu)} \simeq 0$, we get

$$
\frac{m_{\nu_e}}{m_{\nu_\mu}} = \frac{9\,\varepsilon^{(\nu)}}{4\cdot 80} \leftrightarrow \left(\frac{\Delta m^2_{\rm sol}}{\Delta m^2_{\rm atm}}\right)^{1/2} \sim 1.7\times 10^{-4} \;. \qquad (49)
$$

Then,

$$
\varepsilon^{(\nu)} \sim 6.1\times 10^{-3} \qquad (50)
$$

and

$$
\begin{aligned}
m_{\nu_e} &= \frac{\varepsilon^{(\nu)}}{29}\mu^{(\nu)} \sim 2.1\times 10^{-4}\,\mu^{(\nu)} \;, \\
m_{\nu_\mu} = \frac{4\cdot 80}{9\cdot 29}\,\mu^{(\nu)} = 1.2261\,\mu^{(\nu)} \quad &, \quad m_{\nu_\tau} = \frac{24\cdot 624}{25\cdot 29}\,\mu^{(\nu)} = 20.657\,\mu^{(\nu)} = 16.848 m_{\nu_\mu} \;. \qquad (51)
\end{aligned}
$$

Here, the neutrino masses are $m^{I,II}_{\nu_\alpha} = m_{\nu_\alpha}\lambda^{I,II} = m_{\nu_\alpha}(\mp 1 + \lambda^{(M)}) \simeq \mp m_{\nu_\alpha}$, so that $|m^{I,II}_{\nu_\alpha}| \simeq m_{\nu_e}\,,\ m_{\nu_\mu}\,,\ m_{\nu_\tau}\,$, where $m_{\nu_e}\ :\ m_{\nu_\mu}\ :\ m_{\nu_\tau} \sim 1.7 \times 10^{-4} : 1 : 16.8$. From Eqs. (47) and (51) we infer that

$$\mu^{(\nu)\,2}\lambda^{(M)} \sim 3.7 \times 10^{-4}\ \mathrm{eV}^2\ . \tag{52}$$

In this way, both neutrino deficits can be explained by oscillations of unmixed pseudo–Dirac neutrinos ($U^{(\nu)}_{\alpha i} \simeq \delta_{\alpha i}$). Note, however, that now both solar $\nu^{(a)}_e$'s and atmospheric $\nu^{(a)}_\mu$'s oscillate dominantly into Majorana sterile neutrinos: $\nu^{(s)}_e$'s and $\nu^{(s)}_\mu$'s, respectively (in contrast to the previous mixed pseudo–Dirac $\nu^{(a)}_e$ and $\nu^{(a)}_\mu$ neutrinos of which the latter oscillated dominantly into $\nu^{(a)}_\tau$'s). The experimental evidence for $\nu_\mu \to \nu_\tau$ oscillations and/or for the LSND effect would be, of course, crucial in the process of understanding the mechanism of neutrino oscillations.

In the present case, the effective mass $\langle m_{\nu_e}\rangle$ of the Majorana $\nu^{(a)}_e$ neutrino is given as

$$\langle m_{\nu_e}\rangle \simeq m_{\nu_e}\frac{1}{2}\left(\lambda^I + \lambda^{II}\right) = m_{\nu_e}\lambda^{(M)}\ , \tag{53}$$

since $U^{(\nu)}_{\alpha i} \simeq \delta_{\alpha i}$. Thus, the $O\nu\beta\beta$ decay upper bound $\langle m_{\nu_e}\rangle \leq 0.2$ eV is certainly satisfied because of $\lambda^{(M)} \ll 1$ (and $m_{\nu_e} \leq$ a few eV).

If it turned out that both solar ν_e's and atmospheric ν_μ's oscillated into sterile neutrinos, it would not be easy to recognize whether, as discussed above, the latter should be Majorana sterile counterparts of Majorana active ν_e's and ν_μ's, or rather, two extra Dirac sterile neutrinos [7].

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